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A Remark on the Kolmogorov–Stein Inequality*

Ha Huy Bang†

Institute of Mathematics, P.O. Box 631, Bo Ho, 10000 Hanoi, Vietnam

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In this paper, essentially developing the Stein method, we prove the Kolmogorov–Stein inequality for any Orlicz norm (with the same constants). © 1996 Academic Press, Inc.

1. INTRODUCTION

A. N. Kolmogorov has given the following result [1]: Let $f(x)$, $f'(x), \dots, f^{(n)}(x)$ be continuous and bounded on $\mathbb{R}$. Then

$$\|f^{(k)}\|_\infty^n \le C_{k,n}\|f\|_\infty^{n-k}\|f^{(n)}\|_\infty^k,$$

where $0 < k < n$, $C_{k,n} = K_{n-k}^n / K_n^{(n-k)}$,

$$K_i = \frac{4}{\pi}\sum_{j=0}^{\infty} (-1)^j/(2j+1)^{i+1}$$

for even i, while

$$K_i = \frac{4}{\pi}\sum_{j=0}^{\infty} 1/(2j+1)^{i+1}$$

for odd i. Moreover the constants are best possible.

This result has been extended by E. M. Stein to the L_p-norm [2]. The Kolmogorov–Stein inequality and its variants are a problem of interest for

*Supported by the National Basic Research Program in Natural Science and by the NCNST "Applied Mathematics."
† E-mail address: hhbang@thevinh.ac.vn.

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many mathematicians and have various applications (see, for example, [3, 4] and their references).

In this paper, essentially developing the Stein method [2], we prove this inequality for an arbitrary Orlicz norm $\|\cdot\|_\Phi$. The obtained result has been successfully applied to proving the corresponding imbedding theorems [5−7] for Sobolev−Orlicz spaces of infinite order and the result [8] for any Orlicz norm.

2. RESULTS

Let $\Phi(t)\colon [0, +\infty) \to [0, +\infty]$ be an arbitrary Young function [9−12], i.e., $\Phi(0) = 0, \Phi(t) \geq 0, \Phi(t) \equiv 0$, and $\Phi(t)$ is convex. Denote by

$$\overline{\Phi}(t) = \sup_{s \geq 0} \{ts - \Phi(s)\},$$

which is the Young function conjugate to $\Phi(t)$ and $L_\Phi(\mathbb{R})$, the space of measurable functions $u(x)$ such that

$$|\langle u, v\rangle| = \left|\int u(x)v(x)dx\right| < \infty$$

for all $v(x)$ with $\rho(v, \overline{\Phi}) < \infty$, where

$$\rho(v, \overline{\Phi}) = \int \overline{\Phi}(|v(x)|)\, dx.$$

Then $L_\Phi(\mathbb{R})$ is a Banach space with respect to the Orlicz norm

$$\|u\|_\Phi = \sup_{\rho(v, \overline{\Phi}) \leq 1} \left|\int u(x)v(x)\, dx\right|,$$

which is equivalent to the Luxembung norm

$$\|f\|_{(\Phi)} = \inf\left\{\lambda > 0\colon \int \Phi(|f(x)|/\lambda)\, dx \leq 1\right\} < \infty.$$

We have the following results [9]:

LEMMA 1. *Let $u \in L_\Phi(\mathbb{R})$ and $v \in L_{\overline{\Phi}}(\mathbb{R})$. Then*

$$\int |u(x)v(x)|\, dx \leq \|u\|_\Phi \|v\|_{(\overline{\Phi})}.$$

LEMMA 2. *Let $u \in L_\Phi(\mathbb{R})$ and $v \in L_1(\mathbb{R})$. Then*

$$\|u * v\|_\Phi \leq \|u\|_\Phi \|v\|_1.$$

Recall that $\|\cdot\|_\Phi = \|\cdot\|_p$ when $1 \leq p < \infty$ and $\Phi(t) = t^p$; and $\|\cdot\|_\Phi = \|\cdot\|_\infty$ when $\Phi(t) = 0$ for $0 \leq t \leq 1$ and $\Phi(t) = \infty$ for $t > 1$.

THEOREM 1. *Let $\Phi(t)$ be an arbitrary Young function, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $L_\Phi(\mathbb{R})$. Then $f^{(k)}(x) \in L_\Phi(\mathbb{R})$ for all $0 < k < n$ and*

$$\|f^{(k)}\|_\Phi^n \le C_{k,n} \|f\|_\Phi^{n-k} \|f^{(n)}\|_\Phi^k. \tag{1}$$

Proof. We begin to prove (1) with the assumption that $f^{(k)}(x) \in L_\Phi(\mathbb{R})$, $0 \le k \le n$.

Fix $0 < k < n$. It is known that $\rho(v, \overline{\Phi}) = 1$ if and only if $\|v\|_{(\overline{\Phi})} = 1$. Therefore, by the definition we get

$$\|f^{(k)}\|_\Phi = \sup_{\|v\|_{(\overline{\Phi})} \le 1} \left| \int_{-\infty}^{\infty} f^{(k)}(x) v(x)\, dx \right|.$$

Let $\epsilon > 0$. We choose a function $v_\epsilon(x) \in L_{\overline{\Phi}}(\mathbb{R})$ such that $\|v_\epsilon\|_{(\overline{\Phi})} = 1$ and

$$\left| \int_{-\infty}^{\infty} f^{(k)}(x) v_\epsilon(x)\, dx \right| \ge \|f^{(k)}\|_\Phi - \epsilon. \tag{2}$$

Put

$$F_\epsilon(x) = \int_{-\infty}^{\infty} f(x+y) v_\epsilon(y)\, dy.$$

Then $F_\epsilon(x) \in L_\infty(\mathbb{R})$ by virtue of Lemma 1, and

$$F_\epsilon^{(r)}(x) = \int_{-\infty}^{\infty} f^{(r)}(x+y) v_\epsilon(y)\, dy, \qquad 0 \le r \le n. \tag{3}$$

Actually, for every function $\varphi(x) \in C_0^\infty(\mathbb{R})$ it follows from the assumption and Lemma 1 that

$$\begin{aligned}
\langle F_\epsilon^{(r)}(x), \varphi(x) \rangle &= (-1)^r \langle F_\epsilon(x), \varphi^{(r)}(x) \rangle \\
&= (-1)^r \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x+y) v_\epsilon(y)\, dy \right) \varphi^{(r)}(x)\, dx \\
&= (-1)^r \int_{-\infty}^{\infty} v_\epsilon(y) \left(\int_{-\infty}^{\infty} f(x+y) \varphi^{(r)}(x)\, dx \right) dy \\
&= \int_{-\infty}^{\infty} v_\epsilon(y) \left(\int_{-\infty}^{\infty} f^{(r)}(x+y) \varphi(x)\, dx \right) dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f^{(r)}(x+y) v_\epsilon(y)\, dy \right) \varphi(x)\, dx \\
&= \left\langle \int_{-\infty}^{\infty} f^{(r)}(x+y) v_\epsilon(y)\, dy, \varphi(x) \right\rangle.
\end{aligned}$$

So we have proved (3).

For all $x \in \mathbb{R}$, clearly,

$$|F_\epsilon^{(r)}(x)| \leq \|f^{(r)}(x+\cdot)\|_\Phi \|v_\epsilon\|_{(\overline{\Phi})} = \|f^{(r)}\|_\Phi .$$

Now we prove continuity of $F_\epsilon^{(r)}(x)$ on $\mathbb{R}$ $(0 \leq r \leq n)$. We show this for $r = 0$ by contradiction: Assume that for some $\epsilon > 0$, point x^0, and subsequence $|t_k| \to 0$

$$\left| \int_{-\infty}^{\infty} \big(f\big(x^0 + t_k + y\big) - f\big(x^0 + y\big)\big) v_\epsilon(y)\,dy \right| \geq \epsilon, \qquad k \geq 1. \tag{4}$$

Since $f \in L_\Phi$ we get easily $f \in L_{1,loc}(\mathbb{R})$. Then for any $n = 1, 2, \dots,$ $f(t_k + y) \to f(y)$ in $L_1(-n, n)$. Therefore, there exists a subsequence, denoted again by $\{t_k\}$, such that $f(t_k + y) \to f(y)$ a.e. in $(-n, n)$. Therefore, there exists a subsequence (for simplicity of notation we assume that it is coincident with $\{t_k\}$) such that $f(x^0 + t_k + y) \to f(x^0 + y)$ a.e. in $(-\infty, \infty)$.

On the other hand, without loss of generality we may assume that $\rho(2f, \Phi) < \infty$. Therefore by the Young inequality we get

$$\begin{aligned}
&\big| f\big(x^0 + t_k + y\big) - f\big(x^0 + y\big)\big|\, |v_\epsilon(y)| \\
&\quad \leq \Phi\Big(\big| f\big(x^0 + t_k + y\big) - f\big(x^0 + y\big)\big|\Big) + \overline{\Phi}(|v_\epsilon(y)|) \\
&\quad \leq \tfrac{1}{2}\Phi\big(2\big|f\big(x^0 + y\big)\big|\big) + \tfrac{1}{2}\Phi\big(2\big|f\big(x^0 + t_k + y\big)\big|\big) + \overline{\Phi}\big(|v_\epsilon(y)|\big).
\end{aligned}$$

The last expression belongs to $L_1(\mathbb{R})$, therefore by Lebesgue's theorem we have

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} \big| f\big(x^0 + t_k + y\big) - f\big(x^0 + y\big)\big|\, |v_\epsilon(y)|\,dy = 0,$$

which contradicts (4). The cases $1 \leq r \leq n$ are proved similarly. The continuity of $F_\epsilon^{(r)}(x)$ has been proved.

The functions $F_\epsilon^{(r)}(x)$ are continuous and bounded on $\mathbb{R}$. Therefore, it follows from the Kolmogorov inequality and (2)–(3) that

$$\begin{aligned}
\big(\|f^{(k)}\|_\Phi - \epsilon\big)^n &\leq |F_\epsilon^{(k)}(0)|^n \leq \|F_\epsilon^{(k)}\|_\infty^n \\
&\leq C_{k,n} \|F_\epsilon\|_\infty^{n-k} \|F_\epsilon^{(n)}\|_\infty^k .
\end{aligned} \tag{5}$$

On the other hand,

$$\|F_\epsilon\|_\infty \leq \|f(x+y)\|_\Phi \|v_\epsilon(y)\|_{(\overline{\Phi})} = \|f\|_\Phi , \tag{6}$$

$$\|F_\epsilon^{(n)}\|_\infty \leq \|f^{(n)}(x+y)\|_\Phi \|v_\epsilon(y)\|_{(\overline{\Phi})} = \|f^{(n)}\|_\Phi . \tag{7}$$

Combining (5)–(7), we get

$$\left(\|f^{(k)}\|_\Phi - \epsilon\right)^n \le C_{k,n}\|f\|_\Phi^{n-k}\|f^{(n)}\|_\Phi^k.$$

By letting $\epsilon \to 0$ we have (1).

To complete the proof, it remains to show that $f^{(k)} \in L_\Phi(\mathbb{R})$, $0 < k < n$ if $f, f^{(n)} \in L_\Phi(\mathbb{R})$.

Let $\psi_\lambda(x) \in C_0^\infty(\mathbb{R})$, $\psi_\lambda(x) \ge 0$, $\psi_\lambda(x) = 0$ for $|x| \ge \lambda$ and $\int\psi_\lambda(x)dx = 1$. We put $f_\lambda = f * \psi_\lambda$. Then $f_\lambda \in C^\infty(\mathbb{R})$ because of $f \in L_{1,loc}(\mathbb{R})$. Therefore, $f_\lambda^{(k)} = f * \psi_\lambda^{(k)}$, $k \ge 0$, and it is easy to check that $f_\lambda^{(n)} = f^{(n)} * \psi_\lambda$.

On the other hand, it follows from Lemma 2 that $f_\lambda^{(k)} = f * \psi_\lambda^{(k)} \in L_\Phi(\mathbb{R})$, $k \ge 0$. Therefore, by the fact proved above, we have

$$\|f_\lambda^{(k)}\|_\Phi^n \le C_{k,n}\|f_\lambda\|_\Phi^{n-k}\|f_\lambda^{(n)}\|_\Phi^k, \qquad 0 < k < n.$$

Therefore, since

$$\|f_\lambda\|_\Phi \le \|f\|_\Phi \cdot \|\psi_\lambda\|_1 = \|f\|_\Phi, \qquad \|f_\lambda^{(n)}\|_\Phi \le \|f^{(n)}\|_\Phi\|\psi_\lambda\|_1 = \|f^{(n)}\|_\Phi$$

we get that, for any $0 \le k \le n$, the sequence $f_\lambda^{(k)}$ is bounded in $L_\Phi(\mathbb{R})$. Now we prove that, for any $0 \le k \le n$, there exists a subsequence, which is $*$-convergent to some $g_k \in L_\Phi(\mathbb{R})$. (We say that h_λ is $*$-convergent to h, where $h_\lambda, h \in L_\Phi(\mathbb{R})$, if $\int h_\lambda v \to \int hv$ for all $v \in L_{\overline{\Phi}}(\mathbb{R})$.) We will show, for example, the fact that f_λ is $*$-convergent to f by contradiction: Assume that for some $\epsilon_0 > 0$, $v \in L_{\overline{\Phi}}(\mathbb{R})$ and a subsequence $\lambda_k \to 0$,

$$\left|\int \left(f_{\lambda_k}(x) - f(x)\right)v(x)\,dx\right| \ge \epsilon_0, \qquad k \ge 1. \tag{8}$$

Then, it is known that $f_\lambda \to f$, $\lambda \to 0$ in $L_{1,loc}(\mathbb{R})$. Therefore, there exists a subsequence $\{k_m\}$ (for simplicity we assume that $k_m = m$) such that $f_{\lambda_k}(x) \to f(x)$ a.e.

We may assume that $\rho(2f,\Phi) < \infty$. Then it follows from Young's inequality that

$$|f_\lambda(x) - f(x)|\,|v(x)| \le \tfrac{1}{2}\Phi\left(2|f_\lambda(x)|\right) + \tfrac{1}{2}\Phi\left(2|f(x)|\right) + \overline{\Phi}\left(|v(x)|\right),$$

moreover, the right side of the last inequality belongs to $L_1(\mathbb{R})$. Therefore, by virtue of Lebesgue's theorem we get

$$\lim_{k\to\infty}\int \left|f_{\lambda_k}(x) - f(x)\right||v(x)|\,dx = 0$$

because of $f_{\lambda_k}(x) \to f(x)$ a.e., which contradicts (8).

Finally, it follows from $*$-convergence $f_\lambda \to f$ that for any $\varphi \in C_0^\infty(\mathbb{R})$

$$\langle f_\lambda^{(k)}(x), \varphi(x)\rangle = (-1)^k \langle f_\lambda(x), \varphi^{(k)}(x)\rangle \to (-1)^k \langle f(x), \varphi^{(k)}(x)\rangle = \langle f^{(k)}(x), \varphi(x)\rangle.$$

Therefore, since the $*$-convergence of some subsequence of $\{f_\lambda^{(k)}\}$ to $g_k \in L_\Phi(\mathbb{R})$, we get $f^{(k)} = g_k \in L_\Phi(\mathbb{R})$ $(0 < k < n)$. So we have proved the fact that $f^{(k)} \in L_\Phi(\mathbb{R})$ for all $0 < k < n$ if $f, f^{(n)} \in L_\Phi(\mathbb{R})$. The proof is complete.

Remark 1. To obtain Theorem 1 we have developed the Stein method because, for example, the property $[g(x+h) - g(x)]/h \to g'(x)$ in the L_p mean $(1 \le p < \infty)$, which is used in [2], holds for L_Φ only if $\Phi(t)$ satisfies the Δ_2-condition (see [10]).

Remark 2. For periodic functions we have:

THEOREM 2. *Let $\Phi(t)$ be an arbitrary Young function, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $L_\Phi(\mathbb{T})$. Then $f^{(k)}(x) \in L_\Phi(\mathbb{T})$ for all $0 < k < n$ and*

$$|||f^{(k)}|||_\Phi^n \le C_{k,n} |||f|||_\Phi^{n-k} |||f^{(n)}|||_\Phi^k,$$

where $\mathbb{T}$ is the torus and $|||.|||_\Phi$ is the corresponding norm.

Remark 3. By the representation [11, p. 135]

$$\|u\|_{(\Phi)} = \sup_{\|v\|_{\overline{\Phi}} \le 1} \left| \int u(x) v(x)\, dx \right|,$$

it is easy to see that the obtained results still hold for any Luxemburg norm.

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